Exercises

"For example," is not proof.

Jewish Proverb

1. Give two reasons why the set of odd integers under addition is not a group.
2. Referring to Example 13, verify the assertion that subtraction is not associative.
3. Show that $\{1, 2, 3\}$ under multiplication modulo 4 is not a group but that $\{1, 2, 3, 4\}$ under multiplication modulo 5 is a group.
4. Show that the group $GL(2, \mathbf{R})$ of Example 9 is non-Abelian by exhibiting a pair of matrices A and B in $GL(2, \mathbf{R})$ such that $AB \neq BA$.
5. Find the inverse of the element $\begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ in $GL(2, Z_{11})$.
6. Give an example of group elements a and b with the property that $a^{-1}ba \neq b$.
7. Translate each of the following multiplicative expressions into its additive counterpart. Assume that the operation is commutative.
   - **a.** a^2b^3
   - **b.** $a^{-2}(b^{-1}c)^2$
   - **c.** $(ab^2)^{-3}c^2 = e$
8. Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group? Can you see any relationship between this group and $U(8)$?
9. (From the GRE Practice Exam) Let p and q be distinct primes. Suppose that H is a proper subgroup of the integers under addition that contains exactly three elements of the set $\{p, p + q, pq, p^q, q^p\}$. Determine which of the following are the three elements in H.
   - **a.** pq, p^q, q^p
   - **b.** $p + q, pq, p^q$
   - **c.** $p, p + q, pq$
   - **d.** p, p^q, q^p
   - **e.** p, pq, p^q
10. List the members of $H = \{x^2 \mid x \in D_4\}$ and $K = \{x \in D_4 \mid x^2 = e\}$.
11. Prove that the set of all 2×2 matrices with entries from $\mathbf{R}$ and determinant $+1$ is a group under matrix multiplication.
12. For any integer $n > 2$, show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$.